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# A DIFFERENTIAL GAME OF UNLIMITED DURATION*

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The properties of the value function (VF) in a different game of unlimited duration with depreciating performance functional are studied, and two methods of approximating the VF are compared. The VF does not satisfy a Lipschitz condition, due to the type of functional. It is therefore not possible to prove in the general case differential inequalities for the usual directional derivatives. To overcome this difficulty, a generalization of the directional derivative of a continuous function is proposed. It consists in "smearing" the direction by higher-order increments than the increment of the argument. Necessary and sufficient conditions on the VF are obtained in terms of the directional derivative and of the conjugate derivatives. It is shown that the differential inequalities used to find the viscous solutions and the inequalities of the present paper are equivalent in every position. It is also shown that the method of discrete approximation of the stationary Hamilton-Jacobi equation for control problems is likewise applicable for problems of differential game theory. This method is shown to be equivalent to the familar following procedure. Problems of the present type arise e.g., when modelling processes with depreciating performance factor. Such problems of optimal programmed control were previously studied in /1-3/ in the absence of unmonitored noise, which has to be regarded as an opponent player.

**1. Formulation of the problem, and preliminary results.** We consider the controlled system

$$x^{\cdot} = f(x, u, v),\ x \in R^n,\ u \in P \subset R^p,\ v \in Q \subset R^q \tag{1.1}$$

with the performance functional

$$J(x(\cdot), u(\cdot), v(\cdot)) = \int_{t_0}^{+\infty} e^{-\lambda\tau} g(x(\tau), u(\tau), v(\tau))\,d\tau, \quad \lambda > 0 \tag{1.2}$$

Here, $P$ and $Q$ are compacta, the functions $f(\cdot)$ and $g(\cdot)$ are continuous with respect to the set of variables, satisfy a Lipschitz condition with respect to $x$ with constant $L$, and are bounded by the constant $K$; $u$ is the control, $v$ the noise, and $t_0 \in [0, +\infty)$ is the initial instant.

We shall study the game (1.1), (1.2) in the context of the formalization of /4, 5/. We add to system (1.1) the $(n+1)$-th equation

$$y^{\cdot} = \left\|\begin{matrix} x^{\cdot} \\ z^{\cdot}\end{matrix}\right\| = \left\|\begin{matrix} f(x, u, v) \\ e^{-\lambda t} g(x, u, v)\end{matrix}\right\| \tag{1.3}$$

$$y(t_0) = \left\|\begin{matrix} x(t_0) \\ z(t_0)\end{matrix}\right\| = y_0 = \left\|\begin{matrix} x_0 \\ z_0\end{matrix}\right\|$$

$$x \in R^n,\ z \in R^1,\ t \in [t_0, +\infty),\ u \in P,\ v \in Q,\ \lambda > 0$$

which is specified in differential form by functional (1.2). The payoff functional is defined by the relation

$$J^*(y(\cdot)) = \lim_{T\to\infty} z(T) \tag{1.4}$$

Here, $z(T)$ is the value of the $(n+1)$-th coordinate of the motion $y(\cdot)$ of system (1.3) at the instant $T$. Notice that the functionals $J$ of (1.2) and $J^*$ of (1.4) are the same for $z_0 = 0$.

We consider the game (1.3), (1.4) in the classes of first player's strategies $(t, y) \mapsto U(t, y): [0, +\infty) \times R^{n+1} \mapsto P$ and of second player's counter-strategies $(t, y, u) \mapsto V(t, y, u): [0, +\infty) \times R^{n+1} \times P \mapsto Q$. The sets of strategies $U$ and of counter-strategies $V$ are denoted by **U** and **V**

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respectively. On the basis of the results of /4, 5/ we can show that the game (1.3), (1.4) has the value

$$\omega^\circ(t_0, y_0) = \inf_U \sup_{y(\,)} \lim_{T\to+\infty} z(T) = \sup_V \inf_{y(\,)} \lim_{T\to+\infty} z(T) \tag{1.5}$$
$$U \in \mathrm{U},\ V \in \mathrm{V}; \quad y(\cdot) = \left\| \begin{matrix} x(\cdot) \\ z(\cdot) \end{matrix} \right\|$$

The supremum (infimum) with respect to $y(\cdot)$ is calculated in the set $Y(t_0, y_0, U)$ (the set $Y(t_0, y_0, V)$), whose elements are the motions $y(\cdot)$ of system (1.3), generated by strategies $U$ (counter-strategies $V$).

Let us give some properties of the VF $(t, y) \mapsto \omega^\circ(t, y)$.

*Property 1.* Let $\omega_T^\circ: [0, T] \times R^{n+1} \mapsto R^1$ be the VF in a game of finite duration $T$ ($T \in [0, +\infty)$) with dynamic behaviour (1.3) and payoff functional

$$J_T^*(y(\cdot)) = z(T) \tag{1.6}$$

Then,

$$\sup_{(t, y) \in [0, T] \times R^{n+1}} |\omega^\circ(t, y) - \omega_T^\circ(t, y)| \leqslant K\lambda^{-1}e^{-\lambda T} \tag{1.7}$$

*Property 2.* The VF $\omega^\circ$ can be written as

$$\begin{gathered} \omega^\circ(t, y) = \omega^\circ(t, x, z) = z + e^{-\lambda t} w^\circ(x) \\ w^\circ(x) = \omega^\circ(0, x, 0) \\ x \in R^n,\ z \in R^1,\ t \in [0, +\infty) \end{gathered} \tag{1.8}$$

By (1.8), the function $w^\circ: R^n \mapsto R^1$ is the VF of the game

$$x^\cdot = f(x, u, v),\ x(0) = x_0,\ u \in P,\ v \in Q \tag{1.9}$$
$$J_0(x(\cdot), u(\cdot), v(\cdot)) = \int_0^{+\infty} e^{-\lambda\tau} g(x(\tau), u(\tau), v(\tau))\, d\tau, \quad \lambda > 0$$

We shall henceforth consider the function $w^\circ$.

*Property 3.* The VF $w^\circ$ is bounded

$$\sup_{x \in R^n} |w^\circ(x)| \leqslant K\lambda^{-1} \tag{1.10}$$

In view of estimate (1.7), the Lipschitz condition, see /4, 5/ for function $\omega_T^\circ$, and relation (1.8), we have

$$|w^\circ(x_1) - w^\circ(x_2)| \leqslant L\, \|x_1 - x_2\| \int_0^T e^{(L-\lambda)\tau}\, d\tau + K\lambda^{-1}e^{-\lambda T}; \quad x_1, x_2 \in R^n, \quad T \in [0, +\infty) \tag{1.11}$$

*Property 4.* The VF $w^\circ$ is Hölder continuous

$$\sup_{x_1 \neq x_2} \frac{|w^\circ(x_1) - w^\circ(x_2)|}{\|x_1 - x_2\|^\gamma} \leqslant C \tag{1.12}$$
$$\gamma = \begin{cases} 1, & \mu > 1 \\ (0, 1), & \mu = 1, \\ \mu, & \mu < 1 \end{cases} \quad C = \begin{cases} (\mu - 1)^{-1}, & \mu > 1 \\ (\mu(1 - \gamma))^{-1}(K/L)^{1-\gamma}, & \mu = 1 \\ (\mu(1 - \mu))^{-1}(K/L)^{1-\mu}, & \mu < 1, \ \mu = \lambda/L \end{cases}$$

*Note 1.* By estimates (1.12), if $\mu > 1$ the VF $w^\circ$ satisfies a Lipschitz condition. If $\mu \leqslant 1$, examples of games (1.3), (1.4) can be found, i.e., the functions $f(\cdot)$ and $g(\cdot)$ can be chosen, such that $w^\circ$ does not satisfy a Lipschitz condition.

## 2. Differential inequalities.

The most important properties of the VF are the so-called stability ($u$- and $v$-stability) properties, which can be treated as an optimality (sub- or super-optimality) principle of dynamic programming. Differential inequalities were proposed in /6, 7/, which express the stability properties in infinitesimal form for VF which satisfy a Lipschitz condition, in a game of limited duration. Similar inequalities can be obtained for the Hölder continuous VF in the game (1.9) of unlimited duration.

We will introduce some notation and definitions.

We denote by IS the class of functions $r(\cdot): [0, +\infty) \mapsto R^n$ such that $\lim \|r(\delta)\|\, \delta^{-1} = 0$, $\delta \downarrow 0$.

*Definition.* The lower and upper derivatives of the continuous function $w: R^n \mapsto R^1$ at the point $x$ with respect to the direction $h$ are defined respectively by the equations

$$\partial_- w(x)|(h) = \inf_{r(\cdot)\in \mathrm{IS}} \liminf_{\delta\downarrow 0} \Delta w \delta^{-1} \tag{2.1}$$

$$\partial_+ w(x)|(h) = \sup_{r(\cdot)\in \mathrm{IS}} \limsup_{\delta\downarrow 0} \Delta w \delta^{-1}$$

$$\Delta w = w(x + \delta h + r(\delta)) - w(x)$$

(if the function $w$ satisfies a Lipschitz condition, we can put $r(\cdot) \equiv 0$ in (2.1) and omit the operation inf, sup with respect to $r(\cdot) \in \mathrm{IS}$; this definition of the directional derivatives was used in /6-8/.

Let (co $A$ is the convex hull of set $A$)

$$H_1(x, u) = \mathrm{co}\left\{\left\| \begin{matrix} f(x, u, v) \\ g(x, u, v) \end{matrix} \right\| : v \in Q\right\}, \quad u \in P$$

$$H_2(x, v) = \mathrm{co}\left\{\left\| \begin{matrix} f(x, u, v) \\ g(x, u, v) \end{matrix} \right\| : u \in P\right\}, \quad v \in Q$$

$$H(x, l) = \max_{v\in Q} \min_{u\in P} (\langle l, f(x, u, v)\rangle + g(x, u, v))$$

$$x \in R^n, \quad l \in R^n$$

The function $H(\cdot)$ is called the Hamiltonian of game (1.9).

*Theorem 1.* The necessary and sufficient conditions for $w$: $R^n \mapsto R^1$ to be the VF of game (1.9) are:

1) the function $w$ is bounded and Hölder continuous with estimates (1.10), (1.12);

2) for all $x \in R^n$ we have

$$\lambda w(x) \geqslant \sup_v \inf_h (h_2 + \partial_- w(x) \mid (h_1)) \tag{2.2}$$

$$v \in Q, \quad h = \left\| \begin{matrix} h_1 \\ h_2 \end{matrix} \right\| \in H_2(x, v)$$

$$\lambda w(x) \leqslant \inf_u \sup_h (h_2 + \partial_+ w(x) | (h_1))$$

$$u \in P, \quad h = \left\| \begin{matrix} h_1 \\ h_2 \end{matrix} \right\| \in H_1(x, u)$$

or for all $(x, l) \in R^n \times R^n$ we have the equivalent inequalities

$$\sup_{h\in R^n} (\langle l, h\rangle - \partial_- w(x) \mid (h)) \geqslant -\lambda w(x) + H(x, l) \tag{2.3}$$

$$\inf_{h\in R^n} (\langle l, h\rangle - \partial_+ w(x) \mid (h)) \leqslant -\lambda w(x) + H(x, l)$$

The quantities on the left-hand sides of (2.3) are called respectively the upper and lower conjugate derivatives of $w$, calculated at the point $x$ /7/.

*Note 2.* At points where $w$ is differentiable, conditions (2.2) and (2.3) transform into the stationary Hamilton-Jacobi equation

$$-\lambda w(x) + H(x, \mathrm{grad}\, w(x)) = 0 \tag{2.4}$$

If $w$ satisfies a Lipschitz condition, it is differentiable almost everywhere, so that it satisfies Eq.(2.4) almost everywhere. The concept of a viscous solution of Eq.(2.4) was introduced in /9, 10/ and the existence and uniqueness of such a solution were proved. The definition of this concept contains differential inequalities of a different type to (2.2) and (2.3), in terms of the sub- and super-differentials /10/

$$-\lambda w(x) + H(x, d) \leqslant 0, \quad x \in R^n, \quad d \in D_* w(x) \tag{2.5}$$

$$-\lambda w(x) + H(x, d) \geqslant 0, \quad x \in R^n, \quad d \in D^* w(x)$$

(the sets $D_* w(x)$ and $D^* w(x)$ are respectively the sub- and super-differentials of the function $w$ at the point $x$). It can be shown that relations (2.5) are equivalent to inequalities (2.3), i.e., the VF $w^\circ$ of the game (1.9) is the viscous solution of Eq.(2.4).

## 3. Approximation of the value function.

We will consider two methods of approximation. The first is the familiar /5, 11/ following procedure, at each step of which the programmed max-min operator works in some form. The second, the method of discrete approximation of the stationary Hamilton-Jacobi Eq.(2.4), was proposed in /1, 2/ for constructing the function of the optimal result in the control problem ($u(\cdot)$ is the measurable control)

$$\inf\left\{\int_0^{+\infty} e^{-\lambda\tau} g^*(x(\tau), u(\tau))\,d\tau: u(\cdot):[0, +\infty) \mapsto P\right\} \tag{3.1}$$

where $x(\cdot)$ is given by the differential equation

$$x^{\cdot}(t) = f^*(x(t), u(t)),\ t \in [0, +\infty),\ x(0) = x_0 \tag{3.2}$$

The proof of the convergence of the discrete approximation of the equation

$$-\lambda w(x) + \min_{u\in P}(\langle \operatorname{grad} w(x),\ f^*(x, u)\rangle + g^*(x, u)) = 0 \tag{3.3}$$

corresponding to problem (3.1), (3.2) and the estimation of this convergence are based /1, 2/ on the concept of a viscous solution. Since the viscous solution of Eq.(3.3) and the VF $w^\circ$ are, by Note 2, equivalent concepts, the same technique with suitable modifications can be used to obtain similar results for the game (1.9).

Let us discuss these constructions in more detail.

We start with the following procedure. For its application, we use properties 1 and 2 of the VF of game (1.3), (1.4).

From (1.7), (1.8) we have

$$\sup|w^\circ(x) - \omega_T^\circ(0, x, 0)| \leqslant K\lambda^{-1}e^{-\lambda T},\ T \in [0, +\infty) \tag{3.4}$$

Here and throughout, sup is taken with respect to $x \in R^n$.

To approximate the VF $(t, x, z) \mapsto \omega_T^\circ(t, x, z)$: $[0, T] \times R^n \times R^1 \mapsto R^1$, $\omega^\circ(T, x, z) = z$ in the game (1.3), (1.6) with a fixed instant of termination, we shall use the following procedure. It amounts to the following. The time interval $[0, T]$ is divided into $m$ equal parts with step $h > 0$, so that $T = m\cdot h$. At the instant $T$ the approximation $\omega_T^h(T, \cdot, \cdot)$ of the VF $\omega_T^\circ(T, \cdot, \cdot)$ is given by

$$\omega_T^h(T, x, z) = z,\ x \in R^n,\ z \in R^1 \tag{3.5}$$

We next assume that the approximation $\omega_T^h((i+1)h, \cdot, \cdot)$ $(i = 0, \ldots, m-1)$ at the instant $(i+1)h$ has been constructed. To find the approximation $\omega_T^h(ih, \cdot, \cdot)$ we use the relation

$$\begin{gathered}\omega_T^h(ih, x, z) = \max_{v\in Q}\min_{u\in P}\{\omega_T^h((i+1)h, x + hf(x, u, v),\\ z + he^{-\lambda(ih)}g(x, u, v))\}\\ x \in R^n,\ z \in R^1,\ i = 0, \ldots, m-1\end{gathered} \tag{3.6}$$

The operator on the right-hand side of (3.6) is the programmed max-min operator, which is usually employed in the following constructions /5, 11/.

From (3.5), (3.6) we have

$$\omega_T^h(ih, x, z) = z + e^{-\lambda(ih)}w_T^h(ih, x),\qquad i = 0, \ldots, m \tag{3.7}$$

Here,

$$\begin{gathered}\omega_T^h(ih, x) = \max_{v\in Q}\min_{u\in P}\{e^{-\lambda h}w_T^h((i+1)h\\ x + hf(x, u, v)) + hg(x, u, v)\},\ i = 0, \ldots, m-1\end{gathered} \tag{3.8}$$

$$w_T^h(mh, x) = 0,\ x \in R^n \tag{3.9}$$

Using the method for finding estimate (15.1) in /5/, we can prove a theorem for the following procedure (3.5)-(3.9):

*Theorem 2.* The approximation $w_T^h(0, \cdot)$ is uniformly convergent to the VF $\omega_T^\circ(0, \cdot, 0)$ as $h \downarrow 0$. We have the estimate

$$\sup|\omega_T^\circ(0, x, 0) - w_T^h(0, x)| \leqslant h^{1/2}L\int_0^T e^{(L-\lambda)\tau}\,d\tau \tag{3.10}$$

From (3.4) and (3.10) we have

$$\begin{gathered}\sup|w^\circ(x) - w_T^h(0, x)| \leqslant h^{1/2}L\int_0^T e^{(L-\lambda)\tau}\,d\tau + K\lambda^{-1}e^{-\lambda T},\\ T \in [0, +\infty)\end{gathered} \tag{3.11}$$

Inequality (3.11) implies the estimate

$$\sup |w^\circ(x) - w_{T_*}{}^h(0, x)| \leqslant Ch^{\gamma/2} \tag{3.12}$$

where the numbers $\gamma$ and $C$ are given by (1.12), $h \in (0, 1)$, the instant $T_*$ depends on $h$ and is given by the minimum condition for the right-hand side in (3.11).

We now consider the discrete programming method /1, 2 / as applied to the stationary Hamilton-Jacobi Eq.(2.4) corresponding to game (1.9).

The discrete approximation of Eq.(2.4) is defined as the equation

$$-w^h(x) + \max_{v \in Q} \min_{u \in P} ((1 - \lambda h) w^h(x + hf(x, u, v)) + hg(x, u, v)) = 0, \quad x \in R^n, \quad h \in (0, 1/\lambda) \tag{3.13}$$

The next theorem can be proved in the same way as Theorems 2.1, 2.2 of /1/.

*Theorem 3.* Let $h \in (0, 1/\lambda)$. Then (3.13) has a unique solution $w^h$ in the class of Hölder continuous functions. Estimates (1.10) and (1.12) hold for the solution $w^h$.

The proof is based on the fact that the operator $\Pi$ given by

$$\Pi w(x) = \max_{v \in Q} \min_{u \in P} ((1 - \lambda h) w(x + hf(x, u, v)) + hg(x, u, v)) \tag{3.14}$$

is a contraction operator with contraction coefficient $(1 - \lambda h)$. By the principle of contraction mappings, there exists a unique, bounded, and Hölder continuous solution $w^h$ of Eq.(3.13), and an iterative procedure $\bar{w}_p{}^h = \Pi \bar{w}_{p-1}^h$ with initial approximation $\bar{w}_0{}^h$, satisfying conditions (1.10), (1.12), convergent to it uniformly.

In particular, with $\bar{w}_0{}^h \equiv 0$ we have the estimate

$$\sup |w^h(x) - \bar{w}_p{}^h(x)| \leqslant K\lambda^{-1}(1 - \lambda h)^p, \quad p = 1, 2, \ldots \tag{3.15}$$

*Theorem 4.* As $h \downarrow 0$ the solution $w^h$ of Eq.(3.13) is locally uniformly convergent to the viscous solution $w^\circ$ of Eq.(2.4), which, by Note 2, is the VF of game (1.9).

*Theorem 5.* We have the estimate

$$\sup |w^\circ(x) - w^h(x)| \leqslant Bh^{\gamma/2} \tag{3.16}$$

where $h \in (0, \min\{1/\lambda, 1\})$, the Hölder exponent $\gamma$ is given by (1.12), and $B$ is a constant.

From (3.15) and (3.16), with $h \in (0, \min\{1/\lambda, 1\})$ we obtain

$$\sup |w^\circ(x) - \bar{w}_p{}^h(x)| \leqslant Bh^{\gamma/2} + K\lambda^{-1}(1 - \lambda h)^p, \; p = 1, 2, \ldots \tag{3.17}$$

Inequality (3.17) implies the estimate

$$\sup |w^\circ(x) - (w^h{}_{p_*}(x)| \leqslant Gh^{\gamma/2} \tag{3.18}$$

where $\gamma$ is given by (1.12), $G$ is a constant, $h \in (0, \min\{1/\lambda, 1\})$, and the number $p_*$ depends on $h$ and is found from the condition $(1 - \lambda h)^{p_*} \leqslant h^{\gamma/2}$.

Comparing (3.8) and (3.14), and (3.12) and (3.18), we can see that the following procedure and the discrete programming method, while different in form and in the methods of proving the convergence and the estimates, in essence give the same procedure for approximating the VF $w^\circ$ of the game (1.9), with a convergence estimate of order $h^{\gamma/2}$.

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# ON THE EXISTENCE OF AN INTEGRAL INVARIANT OF A SMOOTH DYNAMIC SYSTEM*

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The existence of an integral invariant with a smooth density for a dynamic system in a cylindrical phase space is considered. The well-known Krylov-Bogolyubov theorem guarantees the existence of an invariant measure for any system in a compact space (for a discussion of these topics see /1, 2/). But this measure is often concentrated in invariant sets of small dimensionality and in general is not an integral invariant with a summable density. For useful applications of ergodic theory, and in the theory of the Euler-Jacobi integrating factor, an invariant measure in the form of an integral invariant with smooth density is useful. Effective criteria for the existence of such measures in smooth dynamic systems are described. The general results are illustrated by examples from non-holonomic mechanics.

**1. Formulation of the problem.** Consider the cylindrical phase space $M^n = \mathbf{R}^k \times \mathbf{T}^{n-k}$ with coordinates $x_1, \ldots, x_n$, of which $k$ are linear and $n-k$ angular. Let $v$ be a smooth vector field in $M^n$; the corresponding differential equation is

$$x^{\cdot} = v(x) \tag{1.1}$$

We consider the existence for system (1.1) of the integral invariant

$$\operatorname{mes}(D) = \int_D f(x)\, d^n x \tag{1.2}$$

with smooth positive density $f: M^n \to \mathbf{R}$.

The criterion for the existence of integral invariant (1.2) is the Liouville equation $\operatorname{div}(fv) = 0$, which, since $f$ is positive, can be rewritten as

$$w^{\cdot} = -\operatorname{div} v, \; w = \ln f \tag{1.3}$$

Clearly, $w$ is a smooth function in $M^n$.

By the theorem on the rectification of trajectories, in a small neighbourhood of a non-singular point of system (1.2) there is an entire family of integral invariants. Thus it is worth considering the integral invariant problem either in the neighbourhood of a position of equilibrium, or in a sufficiently large domain of the phase space where the trajectories are reversible.

We know that the equation of motion of holonomic mechanical systems always have a natural invariant measure (the shape of the volume in the space of cotangent fiberings of the space of positions). It was pointed out in /3/ that non-holonomic systems may in general not have an invariant measure with an integrable density.

We will mention two examples of non-holonomic systems which will be used to illustrate our results.

1°. The problem of the rolling of a heavy rigid body over an absolutely rough horizontal plane. Chaplygin found the invariant measure in the case when the surface is bounded by a sphere and the centre of mass of the body is the same as its geometric centre /4/. An invariant measure can also be shown to exist when the rigid body has an axis of symmetry (either geometric or dynamic).

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